

SEQUENCES OF COMMUTATOR OPERATIONS

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ABSTRACT. Given the congruence lattice \mathbb{L} of a finite algebra \mathbf{A} with a Mal'cev term, we look for those sequences of operations on \mathbb{L} that are sequences of higher commutator operations of expansions of \mathbf{A} .

The properties of higher commutators proved so far delimit the number of such sequences: the number is always at most countably infinite; if it is infinite, then \mathbb{L} is the union of two proper subintervals with nonempty intersection.

1. INTRODUCTION

It is well known that for a finite algebra with a Mal'cev term, the isomorphism type of the congruence lattice yields some information on the binary commutator operation. For example, it is well-known that the diamond \mathbb{M}_3 as a congruence lattice forces an algebra \mathbf{A} with Mal'cev term to be abelian, and hence the commutator operation to satisfy $[1, 1]_{\mathbf{A}} = 0$. As a consequence of the results of this note, the congruence lattice of a finite non-nilpotent Mal'cev algebra is equal to the set-theoretic union of two of its proper subintervals; hence congruence lattices that are no such union force the algebra to be nilpotent. This result is obtained by investigating the higher commutator operations as defined by [Bul01]. Given a lattice \mathbb{L} , we will try to delimit the number of sequences $([., .], [., ., .], \dots)$ of operations on \mathbb{L} that could be the sequence of higher commutator operations of some Mal'cev algebra with congruence lattice isomorphic to \mathbb{L} . We will see that the number of such sequences is at most countable, and we characterise when this number is finite.

This is motivated by the search of a classification of finite algebras with a Mal'cev term modulo polynomial equivalence. We call two universal algebras *polynomially equivalent* if they are defined on the same universe and they have the same clone of polynomial operations. For example, for a set M and its power set $P(M)$, the Boolean algebra $(P(M), \cap, \cup, ')$ and the Boolean ring $(P(M), \Delta, \cap)$ are polynomially equivalent since the fundamental operations of each of these two algebras are polynomial operations of the other one. There are several invariants

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of an algebra that depend on the clone of polynomial operations, but not on the particular choice of fundamental operations. One of these invariants is the congruence lattice, expanded with the binary commutator operation introduced in [Smi76], cf. [FM87, MMT87]. Generalizing the binary commutator operation, A. Bulatov introduced multi-placed commutators for an algebra \mathbf{A} [Bul01, Definition 3]. For each $k \in \mathbb{N}$, and each k -tuple $(\alpha_1, \dots, \alpha_k) \in (\text{Con}(\mathbf{A}))^k$, he defined a congruence $[\alpha_1, \dots, \alpha_k]_{\mathbf{A}}$ of \mathbf{A} and named it the *k-ary commutator* of $\alpha_1, \dots, \alpha_k$. When \mathbf{A} has a Mal'cev term, [Mud09, AM10] discuss several properties of these higher commutators. As for the binary term condition commutator, these commutator operations are completely determined by the clone of polynomial functions of an algebra. So with each algebra \mathbf{A} , we can associate the *commutator structure* of \mathbf{A} . This is the structure $(\text{Con}(\mathbf{A}), \wedge, \vee, (f_i)_{i \in \mathbb{N}})$, where $f_i : (\text{Con}(\mathbf{A}))^i \rightarrow \text{Con}(\mathbf{A})$, $(\alpha_1, \dots, \alpha_i) \mapsto [\alpha_1, \dots, \alpha_i]_{\mathbf{A}}$. If \mathbf{A} has a Mal'cev term, then $(\text{Con}(\mathbf{A}), \wedge, \vee)$ is a modular lattice, and for all $n, k \in \mathbb{N}$ with $k \leq n$, we have

- (HC1) $f_n(\alpha_1, \dots, \alpha_n) \leq \alpha_k$.
- (HC2) f_n is monotonous.
- (HC3) $f_{n+1}(\alpha_1, \dots, \alpha_{n+1}) \leq f_n(\alpha_2, \dots, \alpha_{n+1})$.
- (HC4) f_n is symmetric.
- (HC7) f_n is join distributive.
- (HC8) $f_k(\alpha_1, \dots, \alpha_{k-1}, f_{n-k+1}(\alpha_k, \dots, \alpha_n)) \leq f_n(\alpha_1, \dots, \alpha_n)$.

These properties have been stated and established in [Bul01, Mud09, AM10]. We note that the properties (HC5) and (HC6) listed in [AM10] are missing from the list, since they are not properties of the sequence $(f_i)_{i \in \mathbb{N}}$ but rather on the relation of the higher commutators with the underlying algebra. We call two algebras *commutator equivalent* if they have the same commutator structure. Since an algebra has its congruence relations and its higher commutator operations completely determined by the clone of polynomial functions, polynomially equivalent algebras are commutator equivalent. For a converse, it is open whether two finite Mal'cev algebras \mathbf{A} and \mathbf{B} with $\text{Pol}_3(\mathbf{A}) = \text{Pol}_3(\mathbf{B})$ and the same commutator structure must be polynomially equivalent.

Let us now consider an arbitrary sequence $(f_i)_{i \in \mathbb{N}}$ of operations on a lattice \mathbb{L} such that for each $i \in \mathbb{N}$, the function f_i is an i -ary operation on \mathbb{L} . We say the sequence $(f_i)_{i \in \mathbb{N}}$ is *admissible* if it satisfies the list of properties given above. In the present note we will investigate the following problem:

Given a finite modular lattice \mathbb{L} , how many admissible sequences can be defined on \mathbb{L} ?

Hence given the isomorphism type of the congruence lattice of a Mal'cev algebra, we want to delimit the number of possible higher commutator operations on this algebra.

2. THE CARDINALITY OF THE SET OF ADMISSIBLE SEQUENCES

Let \mathbb{L} be a complete lattice, and let $\delta, \varepsilon \in \mathbb{L}$. We say that (δ, ε) is a *splitting pair* of \mathbb{L} if $\delta < 1$, $\varepsilon > 0$, and for all $\alpha \in \mathbb{L}$, we have $\alpha \geq \varepsilon$ or $\alpha \leq \delta$. A splitting pair is called *strong* if $\delta \geq \varepsilon$. The lattice \mathbb{L} *splits* if it has a splitting pair, it *splits strongly* if it has a strong splitting pair.

Let us remark that this splitting property has often arisen in the following context: A splitting pair (δ, ε) in the congruence lattice of an algebra \mathbf{A} is a rich source of congruence preserving operations on \mathbf{A} because every finitary operation that is constant on δ -classes and maps into one ε -class is a congruence preserving function. [Aic02, HMP12] are just two examples in which the splitting property of the congruence lattice was used in this way.

Let $n \in \mathbb{N}$, and let $f : \mathbb{L}^n \rightarrow \mathbb{L}$. Then f is *join distributive* if for all $i \in \{1, \dots, n\}$, and for all $\alpha_1, \dots, \alpha_n \in \mathbb{L}$ and all families $\langle \beta_j \mid j \in J \rangle$ of elements of \mathbb{L} , we have $f(\alpha_1, \dots, \alpha_{i-1}, \bigvee_{j \in J} \beta_j, \alpha_{i+1}, \dots, \alpha_n) = \bigvee_{j \in J} f(\alpha_1, \dots, \alpha_{i-1}, \beta_j, \alpha_{i+1}, \dots, \alpha_n)$. The function f is *symmetric* if $f(\alpha_1, \dots, \alpha_n) = f(\alpha_{\pi(1)}, \dots, \alpha_{\pi(n)})$ for all $\alpha_1, \dots, \alpha_n \in \mathbb{L}$ and $\pi \in S_n$, and f is *monotonous* if it preserves \leq . The sequence $(f_i)_{i \in \mathbb{N}}$ is an *operation sequence on \mathbb{L}* , if for all $i \in \mathbb{N}$, $f_i : \mathbb{L}^i \rightarrow \mathbb{L}$. The operation sequence $(f_i)_{i \in \mathbb{N}}$ is called *admissible* if it satisfies the properties (HC1), (HC2), (HC3), (HC4), (HC7), (HC8).

Theorem 2.1. *Let \mathbb{L} be a finite modular lattice. Then the number of admissible operation sequences on \mathbb{L} is finite if and only if \mathbb{L} does not split strongly, and countably infinite otherwise.*

The proof will be completed at the end of Section 4.

Let $(f_i)_{i \in \mathbb{N}}$ and $(g_i)_{i \in \mathbb{N}}$ be operation sequences on the lattice \mathbb{L} . We say $(f_i)_{i \in \mathbb{N}} \sqsubseteq (g_i)_{i \in \mathbb{N}}$ if for all $i \in \mathbb{N}$ and for all $\alpha_1, \dots, \alpha_i \in \mathbb{L}$, we have $f_i(\alpha_1, \dots, \alpha_i) \leq g_i(\alpha_1, \dots, \alpha_i)$.

Theorem 2.2. *Let \mathbb{L} be a finite lattice, and let S be the set of all admissible operation sequences on \mathbb{L} . Then S is at most countable, and (S, \sqsubseteq) has no infinite descending chains and no infinite antichains.*

This result will be proved in Section 4.

3. PRELIMINARIES ON LATTICES AND ORDERED SETS

By \mathbb{B}_2 , we denote the two element lattice on the set $\{0, 1\}$, and by \mathbb{M}_3 , we denote the diamond. The lattice \mathbb{M}_3 does not split. It is easy to see that the lattices \mathbb{B}_2 and $\mathbb{M}_2 := \mathbb{B}_2 \times \mathbb{B}_2$ split, but do not split strongly. The three element chain $\{0, \theta, 1\}$ splits strongly with splitting pair (θ, θ) .

Lemma 3.1. *Let \mathbb{L} be a modular lattice of finite height that does not split strongly. Then there are $n \in \mathbb{N}_0$ and a lattice \mathbb{M} such that \mathbb{M} does not split and \mathbb{L} is isomorphic to $\mathbb{M} \times (\mathbb{B}_2)^n$.*

Proof: We proceed by induction on the height of \mathbb{L} . If the height is 0, then $|\mathbb{L}| = 1$ and $\mathbb{L} \cong \mathbb{L} \times (\mathbb{B}_2)^0$. Now assume that $|\mathbb{L}| > 1$. If \mathbb{L} does not split, we take $\mathbb{M} := \mathbb{L}$ and $n := 0$. Now assume that \mathbb{L} has a splitting pair (α, β) . We choose an atom ε and a coatom δ of \mathbb{L} with $\delta \geq \alpha$ and $\beta \geq \varepsilon$. Then (δ, ε) is a splitting pair of \mathbb{L} , and since \mathbb{L} does not split strongly, we have $\delta \not\geq \varepsilon$. Let \mathbb{L}_1 be the interval $\mathbb{I}[0, \delta]$, and let $\mathbb{L}_2 := \mathbb{I}[0, \varepsilon]$. By a theorem of Birkhoff [MMT87, Theorem 2.31], the sublattice of \mathbb{L} generated by $L_1 \cup L_2$ is isomorphic to $\mathbb{L}_1 \times \mathbb{L}_2$. But since (δ, ε) is a splitting pair, we have $(x \wedge \delta) \vee (x \wedge \varepsilon) = x$ for all $x \in \mathbb{L}$. To see this, assume $x \leq \delta$. Then $(x \wedge \delta) \vee (x \wedge \varepsilon) = x \vee (x \wedge \varepsilon) = x$. If $x \geq \varepsilon$, then $(x \wedge \delta) \vee (x \wedge \varepsilon) = (x \wedge \delta) \vee \varepsilon = x \wedge (\delta \vee \varepsilon) = x \wedge 1 = x$. Hence the sublattice generated by $L_1 \cup L_2$ is equal to \mathbb{L} . Thus \mathbb{L} is isomorphic to $\mathbb{L}_1 \times \mathbb{L}_2$.

The lattice \mathbb{L}_2 is isomorphic to \mathbb{B}_2 . The lattice \mathbb{L}_1 does not split strongly: suppose $(\delta_1, \varepsilon_1)$ is a strong splitting pair of \mathbb{L}_1 . Then $((\delta_1, \varepsilon), (\varepsilon_1, 0))$ is a strong splitting pair of $\mathbb{L}_1 \times \mathbb{L}_2$, and therefore, \mathbb{L} has a strong splitting pair, a contradiction. Hence applying the induction hypothesis to \mathbb{L}_1 , we obtain a lattice \mathbb{M} that does not split and $n \in \mathbb{N}_0$ with $\mathbb{L}_1 \cong \mathbb{M} \times \mathbb{B}_2^n$, and therefore $\mathbb{L} \cong \mathbb{L}_1 \times \mathbb{B}_2 \cong \mathbb{M} \times \mathbb{B}_2^{n+1}$. \square

Let $\mathbb{A} = (A, \leq)$ be a partially ordered set. We say that \mathbb{A} satisfies the *descending chain condition* if there is no infinitely descending chain $a_1 > a_2 > a_3 > \dots$. The *ascending chain condition* is defined dually. For $m \in \mathbb{N}$, we define a partially ordered set $\mathbb{A}^m = (A^m, \leq)$, where $(a_1, \dots, a_m) \leq (b_1, \dots, b_m)$ if for all $i \in \{1, \dots, m\}$, we have $a_i \leq b_i$. For $\mathbb{A} := (\mathbb{N}, \leq)$, the following lemma is known as Dickson's Lemma [Dic13].

Lemma 3.2 ([Lav76, Lemma 1.2], [AH07, p.195, Example (4)]). *Let \mathbb{A} be a partially ordered set with the descending chain condition and no infinite antichains. Then \mathbb{A}^m satisfies the descending chain condition and has no infinite antichains.*

A subset I of \mathbb{N}_0^m is an *upward closed set* if for all $\mathbf{a} \in I$ and $\mathbf{b} \in \mathbb{N}_0^m$ with $\mathbf{a} \leq \mathbf{b}$, we have $\mathbf{b} \in I$. It is easy to see that every upward closed set U is uniquely determined by its minimal elements. Since the set of minimal elements of U is an antichain, Lemma 3.2 implies that this set is finite. This establishes the following lemma.

Lemma 3.3. *Let $m \in \mathbb{N}$. There are exactly countably infinitely many upward closed subsets of \mathbb{N}_0^m .*

We will also use the following theorem from order theory:

Theorem 3.4 (cf.[AH07, Corollary 4.3],[Mac01, Theorem 1.2]). *Let $m \in \mathbb{N}$, and let \mathcal{L} be the set of upward closed subsets of \mathbb{N}_0^m . Then the partially ordered set (\mathcal{L}, \subseteq) has no infinite antichain and no infinite ascending chain.*

4. SEQUENCES OF OPERATIONS

First, we prove that the set of admissible operation sequences on a finite lattice is at most countable and satisfies certain order properties.

Proof of Theorem 2.2: Let $m := |\mathbb{L}|$, let $\{\alpha_1, \dots, \alpha_m\}$ be the set of elements of \mathbb{L} , and let $F := (f_i)_{i \in \mathbb{N}}$ be an admissible sequence. Then for $(a_1, \dots, a_m) \in \mathbb{N}_0^m \setminus \{(0, \dots, 0)\}$, we define $E(F, (a_1, \dots, a_m))$ by

$$E(F, (a_1, \dots, a_m)) := f_j(\underbrace{\alpha_1, \dots, \alpha_1}_{a_1 \text{ times}}, \dots, \underbrace{\alpha_m, \dots, \alpha_m}_{a_m \text{ times}}),$$

where $j := \sum_{k=1}^m a_k$.

For $\alpha \in \mathbb{L}$, we define $\mathcal{R}_F(\alpha)$ as the subset of \mathbb{N}_0^m given by

$$(4.1) \quad \mathcal{R}_F(\alpha) = \{(a_1, \dots, a_m) \in \mathbb{N}_0^m \setminus \{(0, \dots, 0)\} \mid E(F, (a_1, \dots, a_m)) \leq \alpha\}.$$

Since F is an admissible sequence, $\mathcal{R}_F(\alpha)$ is an upward closed subset of \mathbb{N}_0^m . Let $F = (f_i)_{i \in \mathbb{N}}$ and $G = (g_i)_{i \in \mathbb{N}}$ be two admissible sequences on \mathbb{L} . We will now show that $F \sqsubseteq G$ if and only if for all $\alpha \in \mathbb{L}$, we have $\mathcal{R}_G(\alpha) \subseteq \mathcal{R}_F(\alpha)$. For the “only if”-direction, we let $\alpha \in \mathbb{L}$ and $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{N}_0^m$ such that $\mathbf{a} \in \mathcal{R}_G(\alpha)$. Then $E(G, (a_1, \dots, a_m)) \leq \alpha$, and thus, since $F \sqsubseteq G$, $E(F, (a_1, \dots, a_m)) \leq \alpha$, which implies $(a_1, \dots, a_m) \in \mathcal{R}_F(\alpha)$. For the “if”-direction, we let $k \in \mathbb{N}$ and $\beta_1, \dots, \beta_k \in \mathbb{L}$. Using the symmetry of f_k and g_k , we obtain $(a_1, \dots, a_m) \in \mathbb{N}_0^m$ such that $f_k(\beta_1, \dots, \beta_k) = E(F, (a_1, \dots, a_m))$ and $g_k(\beta_1, \dots, \beta_k) = E(G, (a_1, \dots, a_m))$. From the last equality, we obtain that \mathbf{a} lies in $\mathcal{R}_G(g_k(\beta_1, \dots, \beta_k))$. Hence we have $\mathbf{a} \in \mathcal{R}_F(g_k(\beta_1, \dots, \beta_k))$. Using the symmetry of f_k , this implies $f_k(\beta_1, \dots, \beta_k) \leq g_k(\beta_1, \dots, \beta_k)$. Denoting by \mathcal{U} be the set of upward closed subsets of \mathbb{N}_0^m , we have just proved that (S, \sqsubseteq) is isomorphic to a subset of the dual of $(\mathcal{U}, \subseteq)^m$. Now from Lemma 3.3, we obtain that S is at most countable. By Theorem 3.4, (\mathcal{U}, \subseteq) has no infinite antichain and no infinite ascending chain. Applying Lemma 3.2 to the dual of (\mathcal{U}, \subseteq) , we obtain that $(\mathcal{U}, \subseteq)^m$ satisfies the ascending chain condition and has no infinite antichains. Hence (S, \sqsubseteq) satisfies the descending chain condition and has no infinite antichains. \square

Lemma 4.1. *Let $\mathbb{L}_1, \mathbb{L}_2$ be lattices, let $\mathbb{L} := \mathbb{L}_1 \times \mathbb{L}_2$, and let $(f_i)_{i \in \mathbb{N}}$ be an admissible operation sequence on \mathbb{L} . Then for all $n \in \mathbb{N}$,*

$$(4.2) \quad f_n\left(\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \dots, \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}\right) = f_n\left(\begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \alpha_n \\ 0 \end{pmatrix}\right) \vee f_n\left(\begin{pmatrix} 0 \\ \beta_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \beta_n \end{pmatrix}\right).$$

Proof: We compute $f_n((\frac{\alpha_1}{\beta_1}), \dots, (\frac{\alpha_n}{\beta_n})) = f_n((\frac{\alpha_1}{0}) \vee (\frac{0}{\beta_1}), \dots, (\frac{\alpha_n}{0}) \vee (\frac{0}{\beta_n}))$. Using join distributivity, the last expression is equal to the join of 2^n expressions of the form $f(a_1, \dots, a_n)$ with $a_i \in \{(\frac{\alpha_i}{0}), (\frac{0}{\beta_i})\}$. If $a_i = (\frac{\alpha_i}{0})$ and $a_j = (\frac{0}{\beta_i})$, then by (HC1), $f(a_1, \dots, a_n) \leq a_i \wedge a_j = 0$. Hence $f_n((\frac{\alpha_1}{\beta_1}), \dots, (\frac{\alpha_n}{\beta_n}))$ is equal to the join of the two summands in the right hand side of (4.2) \square

Lemma 4.2. *Let \mathbb{B}_2 be the 2-element lattice. Then there are exactly three admissible operation sequences on \mathbb{B}_2 ; these are $(f_i)_{i \in \mathbb{N}}$, $(g_i)_{i \in \mathbb{N}}$, and $(h_i)_{i \in \mathbb{N}}$ with $f_n(\alpha_1, \dots, \alpha_n) = 0$ for all $n \in \mathbb{N}$, $g_1(1) = 1$ and $g_i = f_i$ for $i \geq 2$, and $h_n(\alpha_1, \dots, \alpha_n) = \alpha_1 \wedge \dots \wedge \alpha_n$ for all $n \in \mathbb{N}$.*

Proof: Let $(s_i)_{i \in \mathbb{N}}$ be an admissible operation sequence on \mathbb{B}_2 . By (HC1), we have $s_i(\alpha_1, \dots, \alpha_n) = 0$ if $0 \in \{\alpha_1, \dots, \alpha_n\}$. Hence we only need to determine $s_i(1, \dots, 1)$. In the case $s_1(1) = 0$, the property (HC3) yields $s_n(1, \dots, 1) \leq s_{n-1}(1, \dots, 1) \leq \dots \leq s_1(1) = 0$ for all $n \in \mathbb{N}$, and thus $(s_i)_{i \in \mathbb{N}} = (f_i)_{i \in \mathbb{N}}$. In the case that $s_1(1) = 1$ and $s_2(1, 1) = 0$, we have $s_n(1, \dots, 1) = 0$ by repeated application of (HC3). In the case $s_1(1) = s_2(1, 1) = 1$, (HC8) yields $s_n(1, \dots, 1) \geq s_{n-1}(1, \dots, 1, s_2(1, 1)) = s_{n-1}(1, \dots, 1)$ for all $n \geq 3$, and thus $(s_i)_{i \in \mathbb{N}} = (h_i)_{i \in \mathbb{N}}$. \square

Lemma 4.3. *Let \mathbb{L} be a finite lattice that does not split, let n be the number of atoms of \mathbb{L} , and let $(f_i)_{i \in \mathbb{N}}$ be an admissible operation sequence on \mathbb{L} . Then for all $k \geq n$, we have $f_k(\gamma_1, \dots, \gamma_k) = 0$ for all $\gamma_1, \dots, \gamma_k \in \mathbb{L}$.*

Proof: Let $\alpha_1, \dots, \alpha_n$ be all the atoms of \mathbb{L} . If $n = 1$, then \mathbb{L} splits. Therefore, $n \geq 2$. For each $i \in \{1, \dots, n\}$, we define $A(i) := \{x \in \mathbb{L} \mid x \not\geq \alpha_i\}$. We first show that for all $i \in \{1, 2, \dots, n\}$, we have $\bigvee \{\alpha \mid \alpha \in A(i)\} = 1$. Let $\delta := \bigvee \{\alpha \mid \alpha \in A(i)\}$. Then for every $x \in L$, we have $x \geq \alpha_i$ or $x \leq \delta$. Hence if $\delta < 1$, then (δ, α_i) is a splitting pair, contradiction the assumptions. Now if $k \geq n$, using (HC3) and (HC7), we obtain

$$\begin{aligned} f_k(1, \dots, 1) &\leq f_n(1, \dots, 1) = f_n\left(\bigvee_{x_1 \in A(1)} x_1, \dots, \bigvee_{x_n \in A(n)} x_n\right) \\ &= \bigvee_{(x_1, \dots, x_n) \in A(1) \times \dots \times A(n)} f_n(x_1, \dots, x_n). \end{aligned}$$

We will now show that each $f_n(x_1, \dots, x_n)$ is equal to 0. Suppose $f_n(x_1, \dots, x_n) > 0$. Then there is an atom $\alpha_j \in L$ with $f_n(x_1, \dots, x_n) \geq \alpha_j$. Hence $\alpha_j \leq f_n(x_1, \dots, x_n) \leq x_j$. This contradicts $x_j \in A(j)$. \square

This Lemma has a consequence on the congruence lattice of a non-nilpotent algebra. An algebra \mathbf{A} with a Mal'cev term is *nilpotent* if and only if its lower central series of congruences defined by $\gamma_1 := 1$, $\gamma_n := [1, \gamma_{n-1}]_{\mathbf{A}}$ for $n \geq 2$ reaches 0 after finitely many steps. We recall that a direct product $\mathbf{B} = \mathbf{A}_1 \times \dots \times \mathbf{A}_n$ is *skew-free* if for every congruence relation β of \mathbf{B} , there are congruences $\alpha_1 \in$

$\text{Con}(\mathbf{A}_1), \dots, \alpha_n \in \text{Con}(\mathbf{A}_n)$ such that for all $(a_1, \dots, a_n), (b_1, \dots, b_n) \in B$, we have $((a_1, \dots, a_n), (b_1, \dots, b_n)) \in \beta$ if and only if $(a_i, b_i) \in \alpha_i$ for all $i \in \{1, \dots, n\}$.

Corollary 4.4. *Let \mathbf{A} be a finite algebra with a Mal'cev term. Then we have:*

- (1) *If \mathbf{A} is not nilpotent, then its congruence lattice $\text{Con}(\mathbf{A})$ splits.*
- (2) *If $\text{Con}(\mathbf{A})$ does not split strongly, then there exist $n \in \mathbb{N}_0$ and algebras $\mathbf{B}, \mathbf{C}_1, \dots, \mathbf{C}_n$ such that \mathbf{A} is isomorphic to the direct product $\mathbf{B} \times \mathbf{C}_1 \times \dots \times \mathbf{C}_n$, \mathbf{B} is nilpotent, each \mathbf{C}_i is simple, and the direct product is skew-free.*

Proof: (1) Assume that the lattice $\text{Con}(\mathbf{A})$ does not split. Then by Lemma 4.3, there is an $n \in \mathbb{N}$ such that the n -ary higher commutator operation of \mathbf{A} satisfies $[1, \dots, 1]_{\mathbf{A}} = 0$. By (HC8) and (HC2), we obtain that then the n -th term γ_n of the lower central series of \mathbf{A} satisfies $\gamma_n = 0$. Hence \mathbf{A} is nilpotent, contradicting the assumptions.

(2) We assume that the congruence lattice of \mathbf{A} does not split strongly. Then Lemma 3.1 yields an $n \in \mathbb{N}_0$ and a lattice \mathbb{M} that does not split such that $\text{Con}(\mathbf{A})$ is isomorphic via some isomorphism ι to $\mathbb{M} \times \mathbb{B}_2^n$. For $i \in \{0, \dots, n\}$, let $\nu_i := \iota^{-1}((1, 1, \dots, 1, 0, 1, \dots, 1))$ with 0 at the $(i+1)$ -th place. Using the fact that these congruences permute, we obtain (cf. [MMT87, p.161]) that \mathbf{A} is isomorphic to $\prod_{i=0}^n (\mathbf{A}/\nu_i)$. Since $\text{Con}(\mathbf{A}/\nu_0)$ is isomorphic to \mathbb{M} , the congruence lattice of \mathbf{A}/ν_0 does not split, and hence, by the first part of this corollary, \mathbf{A}/ν_0 is nilpotent. For $i \geq 1$, ν_i is a coatom of $\text{Con}(\mathbf{A})$ and \mathbf{A}/ν_i is simple. Hence $\mathbf{B} := \mathbf{A}/\nu_0$ and $\mathbf{C}_i := \mathbf{A}/\nu_i$ satisfy $\mathbf{A} \cong \mathbf{B} \times \prod_{i=1}^n \mathbf{C}_i$. For every $\theta \in \text{Con}(\mathbf{A})$, we have $\theta = \iota^{-1}(\iota(\theta)) = \iota^{-1}((\iota(\theta) \vee (0, 1, 1, \dots, 1)) \wedge \dots \wedge (\iota(\theta) \vee (1, 1, 1, \dots, 0))) = (\theta \vee \nu_0) \wedge \dots \wedge (\theta \vee \nu_n)$, and therefore the direct product is skew-free by [BS81, Lemma IV.11.6]. \square

Theorem 4.5. *Let \mathbb{L} be a finite modular lattice, and let S be the set of all admissible sequences on \mathbb{L} . Then S is infinite if and only if \mathbb{L} splits strongly.*

Proof: Let us assume that \mathbb{L} does not split strongly. Then by Lemma 3.1, \mathbb{L} is isomorphic to a direct product $\mathbb{M} \times \mathbb{B}_2^n$ such that \mathbb{M} does not split. Now by Lemmas 4.3 and 4.2, on each of the direct factors, there are only finitely many admissible operation sequences, and thus by Lemma 4.1, S is finite.

If \mathbb{L} splits strongly, then we choose a strong splitting pair (δ, ε) , and we define an operation sequence $(f_i)_{i \in \mathbb{N}}$ by $f_1(\alpha_1) := \alpha_1$ for all $\alpha_1 \in \mathbb{L}$, and for $i \geq 2$, $f_i(\alpha_1, \dots, \alpha_i) := 0$ if there exists an $j \in \{1, \dots, i\}$ with $\alpha_j \leq \delta$, and $f_i(\alpha_1, \dots, \alpha_i) := \varepsilon$ else. Let $g_i(\alpha_1, \dots, \alpha_i) = 0$ for $i \in \mathbb{N}$. Now we show that for each $k \in \mathbb{N}$, the sequence $(h_i^{(k)})_{i \in \mathbb{N}}$ defined by $h_i^{(j)} := f_i$ for $i \leq j$ and $h_i^{(j)} := g_i$ for $i > j$ is an admissible sequence.

We first show that each f_i satisfies (HC1). Supposing that (HC1) fails for some $\alpha_1, \dots, \alpha_i$, we have $f_i(\alpha_1, \dots, \alpha_i) = \varepsilon$ and thus $\alpha_j \not\leq \delta$ for all $j \in \{1, \dots, i\}$.

Thus $\alpha_j \geq \varepsilon$ for all j , and therefore $f_i(\alpha_1, \dots, \alpha_i) \leq \bigwedge_{j=1}^i \alpha_j$. (HC2), (HC3), and (HC4) are immediate consequences of the definitions. Now for join distributivity, having already established (HC1–4), we only need to prove

$$f_i\left(\bigvee_{j \in J} \beta_j, \alpha_2, \dots, \alpha_i\right) \leq \bigvee_{j \in J} f_i(\beta_j, \alpha_2, \dots, \alpha_i)$$

for all families $\langle \beta_j \mid j \in J \rangle$ from \mathbb{L} . Suppose that the right hand side is 0. Then either one of the α_k satisfies $\alpha_k \leq \delta$, implying that the left hand side is 0, or all α_k satisfy $\alpha_k \not\leq \delta$. Then we have $\beta_j \leq \delta$ for all $j \in J$. This implies $\bigvee_{j \in J} \beta_j \leq \delta$, and therefore the left hand side is 0 as well.

In order to prove (HC8) for each sequence $(h^{(j)})_{i \in \mathbb{N}}$, we observe that for $j \leq i-2$, and for every nested expression of the form $f_{j+1}(\alpha_1, \dots, \alpha_j, f_{j-i}(\alpha_{j+1}, \dots, \alpha_i))$, we have $f_{j+1}(\alpha_1, \dots, \alpha_j, f_{j-i}(\alpha_{j+1}, \dots, \alpha_i)) \leq f_{j+1}(\alpha_1, \dots, \alpha_j, \varepsilon) = 0$. \square

Now Theorem 2.1 follows immediately from Theorems 2.2 and 4.5. As a consequence, we give an upper bound on the number of pairwise commutator inequivalent algebras with Mal'cev term on a finite universe.

Corollary 4.6. *Let A be a finite set, let I be an infinite set, let \mathbb{L} be a sublattice of the lattice of equivalence relations on A , and let $(\mathbf{B}_i)_{i \in I}$ be a family of algebras with universe A such that for each $i \in I$, \mathbf{B}_i has a Mal'cev term d_i and $\text{Con}(\mathbf{B}_i) = \mathbb{L}$, and for all $i, j \in I$ with $i \neq j$, \mathbf{B}_i and \mathbf{B}_j are not commutator equivalent. Then $|I| \leq \aleph_0$, and \mathbb{L} is the union of two intervals $I[0, \delta] \cup I[\varepsilon, 1]$ with $0 < \varepsilon \leq \delta < 1$.*

Proof: For each $i \in I$ and $j \in \mathbb{N}$, we define $h_j^{(i)}(\alpha_1, \dots, \alpha_j) := [\alpha_1, \dots, \alpha_j]_{\mathbf{B}_i}$. Since each \mathbf{B}_i has a Mal'cev term, each $(h_j^{(i)})_{j \in \mathbb{N}}$ is an admissible sequence. Since all \mathbf{B}_i are commutator inequivalent, we get an infinite set of admissible sequences. Thus by Theorem 2.1, I is countably infinite and \mathbb{L} splits strongly. \square

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